

Relativistic rapidity as change in musical pitch

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Relativistic rapidity is usually presented as a computational device. As Lévy-Leblond has shown, it is also the velocity that would be imputed by an ideal Newtonian inertial guidance system, taking $c = 1 \text{ neper} = 1$. Here, we show that it can also be interpreted as the change in musical pitch of radiation fore and aft along the direction of motion.

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I. INTRODUCTION

The usual one-dimensional relativistic velocity addition formula is given by

$$\frac{v_{AB}}{c} \boxplus \frac{v_{BC}}{c} = \frac{\frac{v_{AB}}{c} + \frac{v_{BC}}{c}}{1 + \frac{v_{AB}}{c} \cdot \frac{v_{BC}}{c}}. \quad (1)$$

The rapidity α for a velocity v is defined by

$$\tanh\left(\frac{\alpha}{b}\right) = \frac{v}{c}, \quad (2)$$

with b an arbitrary nonzero constant usually chosen to be unity. We find that

$$\alpha_{AB} + \alpha_{BC} = \alpha_{AC} \Leftrightarrow \frac{v_{AB}}{c} \boxplus \frac{v_{BC}}{c} = \frac{v_{AC}}{c}. \quad (3)$$

In other words, relativistic velocity addition in one spatial dimension is equivalent to ordinary addition of the corresponding rapidities. In a companion paper¹, we discuss the use of velocity factors (two-way doppler factors) to simplify calculations.

Velocity factors

$$f = \frac{c+v}{c-v} \quad (4)$$

and their square roots

$$k = \sqrt{\frac{c+v}{c-v}} \quad (5)$$

can be interpreted respectively as two-way and one-way doppler factors. It is natural to seek a physical interpretation for rapidity as well.

Elegantly, Lévy-Leblond has interpreted rapidity as the integral of proper acceleration². Beginning at rest in a certain frame, accelerating arbitrarily along a straight line, one's current rapidity is always the integral of what an ideal accelerometer would measure. If one chooses the arbitrary constant b to be equal to c , the rapidity is the velocity that an ideal Newtonian inertial guidance system would impute.

II. THE CHANGE-IN-PITCH INTERPRETATION OF RAPIDITY

But we can also give another interpretation. From the definition of rapidity, we find that

$$\begin{aligned}
 \alpha &= b \operatorname{arctanh} \frac{v}{c} \\
 &= b \ln \sqrt{\frac{1+v/c}{1-v/c}} \\
 &= b \ln \sqrt{\frac{c+v}{c-v}} \\
 &= b \ln k,
 \end{aligned} \tag{6}$$

where $k = \sqrt{\frac{c+v}{c-v}}$ is the one-way doppler factor for frequencies, for a velocity of approach v between source and observer. Rapidities compose by addition because they are logarithms of doppler factors, and doppler factors compose by multiplication.

Using generic logarithms $\lg(\cdot)$ ³ and taking $b = 1 \text{ neper} = \lg e$,

$$\begin{aligned}
 \alpha &= \ln k \lg e \\
 &= \frac{\lg k}{\lg e} \lg e \\
 &= \lg k.
 \end{aligned} \tag{7}$$

This a very simple relationship, and says that the rapidity so defined is just the logarithmic level of k . We can further write

$$\begin{aligned}
 \alpha &= \frac{\lg(k)}{\lg(2)} \lg(2) \\
 &= \log_2(k) \text{ octaves} \\
 &= 12 \log_2(k) \text{ semitones},
 \end{aligned} \tag{8}$$

for k the doppler factor of frequency shift along the line of motion. Human pitch perception is logarithmic in frequency. If we had the ability to hear electromagnetic radiation, the rapidity α is simply the doppler-induced shift in pitch. The pitch of radiation from the direction boosted toward goes sharp by α , while the pitch from the opposite direction goes flat by α .

III. DISCUSSION

The following are all equivalent one-dimensional kinematic states :

- light from directly ahead is sharp—blueshifted—by one semitone,
- light from directly behind is flat—redshifted—by one semitone,
- proper acceleration integrated with respect to proper time imputes a Newtonian velocity of $c \ln 2^{1/12} \approx 0.057762 c$, but
- the relativistically correct velocity is

$$\frac{(2^{1/12})^2 - 1}{(2^{1/12})^2 + 1} c \approx 0.057698 \tag{9}$$

At nearly 6% of c , the rapidity is one semitone, while the relativistically correct velocity differs from the Newtonian calculated velocity by about one part in a thousand. For many purposes, then, below this speed one can treat the rapidity and velocity as proportional, and even identify them. On the other hand, the semitone is still a large rapidity compared to those usual for macroscopic terrestrial objects. Taking the fine structure constant as $1/137$ and the mass of electron to be negligible compared to that of the proton, then the velocity of an orbital electron in a hydrogenic atom is $c/137$. This velocity corresponds to a rapidity of about an eighth of a semitone. The cent, a pitch unit used in piano tuning and defined as one percent of a semitone, corresponds to a velocity of

$$\frac{2^{2/1200} - 1}{2^{2/1200} + 1} c \approx \ln 2 \cdot c/1200 \approx 173 \text{ km/s}, \quad (10)$$

still a very large velocity for macroscopic terrestrial objects. For comparison, the orbital velocity of the sun about the galactic center is about 217 km/s, escape velocity at the surface of the sun is about 618 km/s, and escape velocity from Jupiter's surface is about 60 km/s.

A rapidity of one millionth of a semitone corresponds to a speed of about 17.32 m/s, which is just over 60 km/h, or just under 40 miles/hour. Highway patrol radar guns need to resolve frequency differences to at least an order of magnitude smaller, or a ten millionth of a semitone.

One nanosemitone corresponds to about 1.732 cm/s. Still, these quantities differ in much the way that a small angle differs from its tangent. Rapidities add in a relativistically correct fashion, while velocities do not, except approximately. If human vision had such extreme frequency resolution that we could see doppler effects at familiar speeds, or if we could somehow hear light to such resolution, perhaps an intuitive understanding of relativity would be much easier to develop.

¹ A. T. Wilson, American Journal of Physics (????).

² J. Marc Lévy-Leblond, American Journal of Physics **48**, 345 (1980).

³ A. T. Wilson (in preparation).